ELSEVIER Journal of Computational and Applied Mathematics 57 (1995) 271–281

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

A large family of semi-classical polynomials: The perturbed Chebyshev

Gabriela Sansigre[a], Galliano Valent[b,*]

[a]*Dpto. Matemática Aplicada, E.T.S. Ingenieros Industriales, c/ José Gutiérrez Abascal 2, 28006 Madrid, Spain*
[b]*Laborataire de Physique Théorique et Hautes Energies–CNRS, Université Paris VII, Tour 24, 5e ét., 2 Place Jussieu, 75251 Paris Cedex 05, France*

Received 23 October 1992; revised 2 May 1993

Abstract

Some results concerning finite perturbations of order s of the Chebyshev polynomials are shown. Their semi-classical character follows from the Stieltjes transform of their orthogonality measure. According to the choice of the perturbation parameters, polynomials of arbitrary class can be generated. The second-order differential equation satisfied by these polynomials is obtained and several examples are worked out. Some particular cases of this differential equation had already been obtained by Shohat and Allaway.

Keywords: Semi-classical polynomials; Orthogonality measure; Second-order differential equation

1. Introduction

A sequence of orthogonal polynomials $P_n(x)$ with three-term recurrence relation, and positive-definite orthogonality, is said to be semi-classical (SC) if the Stieltjes transform of their orthogonality measure satisfies the first-order differential equation

$$A(z)S'(z) = C(z)S(z) + D(z), \quad S(z) = \int_{-\infty}^{+\infty} \frac{d\mu(s)}{z - s}, \qquad z \in \mathbb{C}\backslash\mathbb{R}, \tag{1}$$

where A, C, D are *irreducible* polynomials, i.e., with no common factor. It follows from [6] that the orthogonality measure is *unique* and therefore the corresponding moment problem is determined.

The word semi-classical was coined up in [6] and systematically used after [11], but these polynomials had been thoroughly studied in [7, 16, 17].

* Corresponding author. e-mail: valent@lpthe.jussieu.fr.

SSDI 0377-0427(93)E0251-G

The class of the SC polynomials is then defined [11] by

$$c = \max(\deg C - 1, \deg A - 2) \tag{2}$$

and the classical polynomials (Hermite, Laguerre, Bessel, Jacobi) correspond to $c = 0$.

It has been proved in [12] that relation (1) implies, for the absolutely continuous part of the orthogonality measure $\mu\,dx$, the relation

$$A\mu' = C\mu. \tag{3}$$

Laguerre, in a pioneering article [7] obtained the most important properties of the SC polynomials among which stems the existence of a second-order differential equation with polynomial coefficients. Later on, numerous authors have given different derivations of this equation and worked out many examples [6, 10, 11, 15–17].

In this paper we shall consider all the finite perturbations of order $s \geqslant 2$ of the second kind Chebyshev defined by the recurrence

$$\begin{aligned} &P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x), \quad n \geqslant s,\\ &P_{n+1}(x) = 2(x - b_n)P_n(x) - a_nP_{n-1}(x), \quad s - 1 \geqslant n \geqslant 1,\\ &P_0(x) = 1, \qquad P_1(x) = 2(x - b_0), \end{aligned} \tag{4}$$

where the initial conditions are linked to the array of parameters

$$\begin{pmatrix} b_0 & b_1 & \cdots & b_{s-1} \\ & a_1 & \cdots & a_{s-1} \end{pmatrix}$$

with real b_n and strictly positive a_n.

In Section 2 we have gathered background material which includes a short summary of Laguerre general results on the SC polynomials. We then specialize to be perturbed Chebyshev for which we review the results of Geronimus on their Stieltjes function $S(z)$ and the absolutely continuous part of their orthogonality measure. We prove their SC character and also that they share with their unperturbed partners a simple property: their Stieltjes function $S(z)$ is solution of the second degree equation

$$E(z)S^2(z) + F(z)S(z) + G(z) = 0$$

with polynomials $E(z)$, $F(z)$, $G(z)$. We also give an upper bound for the class c which may take any positive integer value, according to the choice of the parameters array.

We conclude by giving an explicit representation formula for the $P_n(x)$ in terms of the second kind Chebyshev $U_n(x)$ and of the initial conditions $(P_0, \dots, P_s)$, and use it to prove the structural relations.

In Section 3, using the same representation formula for the polynomials, we give a direct derivation of a second-order differential equation which is definitely simpler than the one given by Laguerre's treatment.

In Section 4 we give several examples of increasing class. The simplest one, of class 1, deals with the co-recursive second kind Chebyshev studied in [2]. For the class 2 we retrieve a situation studied in [1] and in a particular case in [17]. We conclude with a genuinely new symmetric example of class $2s - 2$ with $s \geqslant 2$.

2. Background material

In view of its intrinsic interest and beauty we give a summary of Laguerre's results on the SC polynomials using modern notations. The reader is referred to [7, 8].

Starting with the monic polynomials

$$M_{n+1}(x) = (x - \beta_n)M_n(x) - \gamma_n M_{n-1}(x), \quad n \geqslant 1,$$
$$M_0(x) = 1, \qquad M_1(x) = x - \beta_0 \tag{5}$$

with $\gamma_n > 0$, Laguerre proves the so-called "structural relations" the first of which can be written

$$AM_n'(x) = \tfrac{1}{2}(C_n - C)M_n(x) + \gamma_n D_n(x) M_{n-1}(x), \quad n \geqslant 0, \qquad M_{-1} \equiv 0, \tag{6}$$

where C_n and D_n are polynomials related by the recurrences

$$C_{n+1} = -C_n - 2(x - \beta_n)D_n,$$
$$\gamma_{n+1}D_{n+1} = A + \gamma_n D_{n-1} + (x - \beta_n)C_n + (x - \beta_n)^2 D_n, \quad D_{-1} \equiv 0, \tag{7}$$

valid for $n \geqslant 0$. We take $C_0 = C$, $D_0 = D$, where C and D are defined in relation (1).

Combined use of (7) and of the recurrence relation (5) gives from (6) the second structural relation

$$AM_n'(x) = -\tfrac{1}{2}(C_{n+1} + C)M_n(x) - D_n M_{n+1}(x), \quad n \geqslant 0. \tag{8}$$

The structural relations (6) and (8) are the essential tools to get the second-order differential equation for the M_n which reads

$$AD_n M_n'' + [(A' + C)D_n - AD_n']M_n' + [\tfrac{1}{2}(C_n - C)D_n' - \tfrac{1}{2}(C_n - C)'D_n + D_n S_{n-1}]M_n = 0 \tag{9}$$

with the notation

$$S_n = \sum_{l=0}^{n} D_l, \qquad S_{-1} \equiv 0.$$

All what is needed for any concrete example is to compute the functions C_n, D_n which appear in the structural relations.

Let us now turn ourselves to the study of the finite perturbations of the Chebyshev polynomials. Their Stieltjes function and the absolutely continuous part of their orthogonality measure have been derived by several authors, the first of whom is Geronimus [5, p. 52]. Later on Dombrowski and Nevai [4], studying infinite perturbations of the Chebyshev, recovered it again as a particular case. More recently Nevai and Van Assche [13] have studied compact perturbations of orthogonal polynomials and (14) follows from their relation (5.3). The most recent derivation is due to Peherstorfer [14, Theorem 3.9].

We need the associated polynomials of order one $P_n^{(1)}$ defined by

$$P_{n+1}^{(1)}(x) = 2xP_n^{(1)}(x) - P_{n-1}^{(1)}(x), \quad n \geqslant s - 1,$$
$$P_{n+1}^{(1)}(x) = 2(x - b_{n+1})P_n^{(1)}(x) - a_{n+1}P_{n-1}^{(1)}(x), \quad s - 2 \geqslant n \geqslant 1, \tag{10}$$
$$P_0^{(1)}(x) = 1, \qquad P_1^{(1)}(x) = 2(x - b_1),$$

whose array is

$$\begin{pmatrix} b_1 & b_2 & \cdots & b_{s-2} & 0 \\ & a_2 & \cdots & a_{s-2} & 1 \end{pmatrix} \tag{11}$$

and a set of four polynomials defined by

$$\mathscr{A}(z) = U_s(z)P_s^{(1)}(z) - U_{s+1}(z)P_{s-1}^{(1)}(z), \qquad \mathscr{C}(z) = U_s(z)P_{s-1}^{(1)}(z) - U_{s-1}(z)P_s^{(1)}(z),$$

$$\mathscr{B}(z) = U_{s-1}(z)P_s(z) - U_s(z)P_{s-1}(z), \qquad \mathscr{D}(z) = U_{s-1}(z)P_{s-1}(z) - U_{s-2}(z)P_s(z).$$

The Stieltjes function of the unperturbed second kind Chebyshev is

$$\tfrac{1}{2} S_0(z) = z - \sqrt{z^2 - 1}, \tag{12}$$

whose branch is real and smaller than 1 for real values of $z \geqslant 1$.

We have the following result.

Theorem 2.1 (Geronimus [5]). *The Stieltjes function of the perturbed Chebyshev is given by*

$$\frac{S(z)}{2} = \frac{\mathscr{A}(z)\frac{1}{2}S_0(z) + \mathscr{C}(z)}{\mathscr{B}(z)\frac{1}{2}S_0(z) + \mathscr{D}(z)}, \quad \mathscr{A}\mathscr{D} - \mathscr{B}\mathscr{C} = a_1 \cdots a_{s-1} > 0 \tag{13}$$

for $z \in \mathbb{C}\backslash\mathbb{R}$ and the absolutely continuous part of the orthogonality measure is

$$\mu = \frac{a_1 \cdots a_{s-1}}{\Delta_s}\mu_0, \quad \mu_0 = \frac{2}{\pi}\sqrt{1 - x^2}\,\mathrm{d}x, \tag{14}$$

where $\Delta_s(x) = P_s^2(x) - 2xP_s(x)P_{s-1}(x) + P_s^2(x)$.

Remark 2.2. It is easy to check that for the array

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ & 1 & \cdots & 1 \end{pmatrix}, \tag{15}$$

we are indeed back to the orthogonality measure of the unperturbed Chebyshev. Furthermore relations (13) and (14) remain valid in the case $s = 1$ provided that in (14) we replace $a_1 \cdots a_{s-1}$ by one.

Remark 2.3. One should observe that the denominator Δ_s can be written

$$\Delta_s = (P_s - xP_{s-1})^2 + (1 - x^2)P_{s-1}^2$$

on which we see that $\Delta_s > 0$ in $]-1, +1[$ and may vanish at ± 1 and possibly for $|x| > 1$. That is, we can have (in fact, we have) isolated poles, but they do not affect the derivation of the differential equation.

The SC character of the perturbed Chebyshev polynomials is easy to ascertain using (13). Indeed Eq. (3) holds with

$$A(x) = (1 - x^2)\Delta_s(x), \qquad C = -(1 - x^2)\Delta_s'(x) - x\Delta_s(x).$$

From relation (2) we see that the class is $c = \deg \Delta_s$ provided that A, C and D are irreducible. If this is not the case this means that they have a common factor which is a polynomial of degree $p \geqslant 1$; in such a case the class is lowered to $\deg \Delta_s - p$. It is not so easy to give the precise value of c for a given array because, as will be apparent on the examples of Section 4, these values are sensitively dependent on the parameters choice. However, for a given array the maximal value of the class c is easy to determine. We have found

$$0 \leqslant c\begin{pmatrix} b_0 & b_1 & \cdots & b_{s-1} \\ & a_1 & \cdots & a_{s-1} \end{pmatrix} \leqslant 2s-1, \quad b_{s-1} \neq 0$$

and

$$0 \leqslant c\begin{pmatrix} b_0 & b_1 & \cdots & 0 \\ & a_1 & \cdots & a_{s-1} \end{pmatrix} \leqslant 2s-2, \qquad a_{s-1} \neq 1.$$

We shall now prove that the perturbed Chebyshev measure is a *second-degree measure* [12] i.e., its Stieltjes transform is the solution of a second-degree equation with polynomial coefficients. To get this equation we invert relation (13) to

$$\frac{S_0(z)}{2} = \frac{\mathscr{D}\frac{1}{2}S - \mathscr{C}}{-\mathscr{B}\frac{1}{2}S + \mathscr{A}}$$

and impose the second-degree equation verified by $S_0(z)$ which is a consequence of (12):

$$(\tfrac{1}{2}S_0(z))^2 - 2z(\tfrac{1}{2}S_0(z)) + 1 = 0.$$

This gives a second-order equation which can be written

$$\Delta_s(\tfrac{1}{2}S(z))^2 + \Lambda_s(z)(\tfrac{1}{2}S(z)) + \Delta_s^{(1)} = 0$$

with

$$\Delta_s^{(1)}(z) = \mathscr{A}^2 + 2z\mathscr{A}\mathscr{C} + \mathscr{C}^2 = (P_{s-1}^{(1)})^2 - 2zP_{s-1}^{(1)}P_s^{(1)} + (P_s^{(1)})^2$$

and the polynomial character of Δ_s, Λ_s and $\Delta_s^{(1)}$ is easily ascertained.

To conclude this section let us derive the structural relations. Noticing that here we have $M_n(x) = 2^{-n}P_n(x)$ the second structural relation becomes

$$AP_n' = -\tfrac{1}{2}(C_{n+1} + C)P_n - \tfrac{1}{2}D_nP_{n+1}, \quad n \geqslant 0. \tag{16}$$

In order to get the polynomials C_n and D_n we use the relation

$$P_n(x) = U_{n-s}(x)P_s(x) - U_{n-s-1}(x)P_{s-1}(x), \quad n \geqslant s, \qquad U_{-1} \equiv 0, \tag{17}$$

which can be proved inductively using the recurrence relation (4).

Relation (17) implies

$$P_{n+1} = -P_sU_{n-s-1} + (2xP_s - P_{s-1})U_{n-s}$$

and inverting the last two relations gives

$$\Delta_sU_{n-s-1} = (2xP_s - P_{s-1})P_{n-1} - P_sP_{n+1}, \qquad \Delta_sU_{n-s} = P_sP_n - P_{s-1}P_{n+1}. \tag{18}$$

Using (17) one can prove

$$(1-x^2)P_n' = -[(1-x^2)P_{s-1}' + (n-s+1)(xP_{s-1} - P_s)]U_{n-s-1} + [(1-x^2)P_s' + (n-s)(P_{s-1} - xP_s)]U_{n-s}.$$

We multiply both members of this equality by Δ_s and get rid of the U's using identities (18). This gives the structural relation (16) with the identifications

$$\tfrac{1}{2}D_n = (1-x^2)(P_{s-1}P_s' - P_sP_{s-1}') + P_s^2 - xP_{s-1}P_s + (n-s)\Delta_s \tag{19}$$

and

$$-\tfrac{1}{2}(C_{n+1} + C) = (1-x^2)[(P_{s-1} - 2xP_s)P_{s-1}' + P_sP_s'] + P_s(xP_s - P_{s-1}) + (n-s+1)x\Delta_s,$$

so that we have proved the structural relation (6) for $n \geqslant s$. For $0 \leqslant n < s$ the structural relations are still valid from Laguerre's theory, but there is no simple way to write C_n and D_n since they do depend on the choice of the parameter array. One has to compute $(P_0 \cdots P_s)$ via the recurrence relation and then use relation (8).

3. Derivation of the differential equation

The change of variable $x = \cos\theta$ brings relation (17) to the form

$$P_n = \alpha(\theta)\cos n\theta + \beta(\theta)\sin n\theta, \tag{20}$$

where α and β given by

$$\alpha(\theta) = -\frac{\sin(s-1)\theta}{\sin\theta}P_s + \frac{\sin s\theta}{\sin\theta}P_{s-1}, \qquad \beta(\theta) = \frac{\cos(s-1)\theta}{\sin\theta}P_s - \frac{\cos s\theta}{\sin\theta}P_{s-1}.$$

These quantities are related to those defined in the previous section by

$$\alpha = \mathscr{D}(\cos\theta), \qquad \sin\theta\beta = \cos\theta\,\mathscr{D}(\cos\theta) + \mathscr{B}(\cos\theta).$$

From these definitions and from (19) it follows that

$$\Delta_s = \sin^2\theta(\alpha^2 + \beta^2), \quad \tfrac{1}{2}D_n = \sin^2\theta[\beta\partial_\theta\alpha - \alpha\partial_\theta\beta + n(\alpha^2 + \beta^2)]. \tag{21}$$

We are now in position to derive a first useful identity; for this we differentiate (20)

$$\partial_\theta P_n = (\partial_\theta\alpha + n\beta)\cos n\theta + (\partial_\theta\beta - n\alpha)\sin n\theta \tag{22}$$

and remove the $\cos n\theta$ with (20)

$$\alpha\sin\theta\partial_\theta P_n - \sin\theta(\partial_\theta\alpha + n\beta)P_n = \sin\theta[\alpha\partial_\theta\beta - \beta\partial_\theta\alpha - n(\alpha^2 + \beta^2)]\sin n\theta.$$

Using (21) and writing derivatives with respect to x (with the notation "prime"), we get

$$(1-x^2)(\alpha P_n' - \alpha' P_n) + n\gamma P_n = -\tfrac{1}{2}D_nU_{n-1}, \quad \gamma = T_{s-1}P_s - T_sP_{s-1}. \tag{23}$$

We need another identity involving second derivatives. For this we differentiate (22) once more

$$\partial_\theta^2 P_n = (\partial_\theta^2\alpha + 2n\partial_\theta\beta - n^2\alpha)\cos n\theta + (\partial_\theta^2\beta - 2n\partial_\theta\alpha - n^2\beta)\sin n\theta$$

and combine this relation with (22) to get

$$\partial_\theta^2 P_n + 2\frac{\cos\theta}{\sin\theta}\partial_\theta P_n = G_n \cos n\theta + H_n \sin n\theta, \tag{24}$$

where

$$\begin{aligned} G_n &= \partial_\theta^2 \alpha + 2n\partial_\theta \beta - n^2\alpha + 2\frac{\cos\theta}{\sin\theta}(\partial_\theta\alpha + n\beta), \\ H_n &= \partial_\theta^2 \beta - 2n\partial_\theta \alpha - n^2\beta + 2\frac{\cos\theta}{\sin\theta}(\partial_\theta\beta - n\alpha). \end{aligned} \tag{25}$$

Removing in (24) $\cos n\theta$ upon use of (20) gives

$$\alpha\left[\partial_\theta^2 P_n + 2\frac{\cos\theta}{\sin\theta}\partial_\theta P_n\right] - G_n P_n = \sin\theta(\alpha H_n - \beta G_n)\frac{\sin n\theta}{\sin\theta} \tag{26}$$

and from (25) it is easy to check that

$$\sin\theta(\alpha H_n - \beta G_n) = -\tfrac{1}{2}D'_n.$$

Coming back to the variable x brings (26) to

$$\alpha[(1-x^2)P''_n - 3xP'_n] - [(1-x^2)\alpha'' - 3x\alpha']P_n + (n^2\alpha + 2n\gamma')P_n = -\tfrac{1}{2}D'_n U_{n-1}. \tag{27}$$

Now we combine (23) and (27) to get the final form of the differential equation for $n \geqslant s$

$$\begin{aligned} &D_n\{\alpha[(1-x^2)P''_n - 3xP'_n] - [(1-x^2)\alpha'' - 3x\alpha']P_n + (n^2\alpha + 2n\gamma')P_n\} \\ &\quad = D'_n\{\alpha(1-x^2)P'_n - (1-x^2)\alpha' P_n + n\gamma P_n\}. \end{aligned} \tag{28}$$

For a given example we just need to compute the quantities $\alpha, \gamma, \Delta_s, D_n$ using the initial conditions $(P_0 \cdots P_s)$.

It is interesting, in (28) to examine the coefficient of P''_n which appears to be $(1-x^2)\alpha D_n$ while from the general results of Laguerre (see Section 2) we would have expected $AD_n \equiv (1-x^2)\Delta_s D_n$.

As will be apparent from the examples, $\alpha(x)$ is simpler than $\Delta_s(x)$! This is the main reason why we have worked out an independent derivation of the differential equation which gives the simplest form for it.

In order to understand the relation between our differential equation and Laguerre's one we get rid of $P''_n(x)$ in (9) using our Eq. (28). Taking into account the relations

$$A = (1-x^2)\Delta_s, \qquad A' + C = -3x\Delta_s$$

one is left with a nontrivial identity from which the polynomials P_n are absent

$$\begin{aligned} &D_n\{\tfrac{1}{2}\alpha(C_n - C)' - \alpha S_{n-1} - [(1-x^2)\alpha'' - 3x\alpha']\Delta_s + n(n\alpha + 2\gamma')\Delta_s\} \\ &\quad = D'_n\{\tfrac{1}{2}\alpha(C_n - C) - [(1-x^2)\alpha' - n\gamma]\Delta_s\}, \quad n \geqslant s \end{aligned} \tag{29}$$

and whose very existence explains that Laguerre's equation can be reduced to the simpler form (28). The reader should observe that there is no indication in Laguerre's framework on this simplified form of the differential equation, which is probably specific to the perturbed Chebyshev.

4. Examples

4.1. Class 1

The recurrence relation is

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x), \quad n \geqslant 1,$$
$$P_0(x) = 1, \qquad P_1(x) = 2(x - b_0).$$

In this case we have the absolutely continuous part of the orthogonality measure

$$\mu = \frac{\mu_0}{1 + b^2 - 2bx}, \qquad \mu_0 = \frac{2}{\pi}\sqrt{1 - x^2}\,\mathrm{d}x, \qquad b = 2b_0 \tag{30}$$

in agreement with the result in [2].

Simple calculations lead to

$$\alpha = 1, \quad \gamma = x - b, \quad \Delta_1 = 1 + b^2 - 2bx, \quad \tfrac{1}{2}D_n = 1 - bx + n(1 + b^2 - 2bx)$$

and to the differential equation:

$$D_n\{(1 - x^2)P_n'' - 3xP_n' + n(n + 2)P_n\} = D_n'\{(1 - x^2)P_n' + n(x - b)P_n\}.$$

One should notice that the polynomials $P_n(b = \pm 1; x)$ are related to the Jacobi polynomials since we have

$$\mu_+ = \frac{1}{\pi}\sqrt{\frac{1 + x}{1 - x}}\,\mathrm{d}x, \qquad \mu_- = \frac{1}{\pi}\sqrt{\frac{1 - x}{1 + x}}\,\mathrm{d}x.$$

The precise link is given by [9, p. 211]

$$P_n(b; x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-b/2, b/2)}(x), \quad b = \pm 1.$$

The class is therefore lowered to zero for three particular values of the parameter: $b = 0, \pm 1$.

In [12] it is claimed that all the *classical* measures of second degree (see Section 2 for the definition) are given by the Jacobi ones with the parameters

$$(\tfrac{1}{2}, \tfrac{1}{2}), \qquad (-\tfrac{1}{2} - n, -\tfrac{1}{2} + n), \quad n \in \mathbb{Z}.$$

This enumeration is certainly not complete since our example shows that the Jacobi polynomials with parameters $(\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$ give rise to classical measures of the second degree.

4.2. Class 2

In the recurrence relation (4) we take the array of parameters

$$\begin{pmatrix} b_0 & 0 \\ & a_1 \end{pmatrix}.$$

In this situation we have

$$\alpha = a_1, \quad \gamma = (2-a_1)x - 2b_0, \quad \Delta_2 = a_1^2 + 4b_0^2 - 4b_0(2-a_1)x + 4(1-a_1)x^2,$$
$$\tfrac{1}{2}D_n = a_1(2-a_1) - 2a_1b_0x + n\Delta_2$$

and the absolutely continuous part of the orthogonality measure is

$$\mu = \frac{a_1}{\Delta_2}\mu_0. \tag{31}$$

This case was first studied by Allaway in his thesis [1], where he derived both the orthogonality measure and the second-order differential equation. He defined the polynomials $\tilde{P}_n(s, r; x)$ by the recurrence

$$\tilde{P}_{n+1}(x) = 2x\tilde{P}_n(x) - \tilde{P}_{n-1}(x), \quad n \geqslant 2,$$
$$\tilde{P}_2(x) = 2x\tilde{P}_1(x) - \tilde{P}_0(x),$$
$$\tilde{P}_0(x) = 1, \qquad \tilde{P}_1(x) = 2sx - r, \quad s \neq 0,$$

which are related to ours by the simple transformation

$$\tilde{P}_0(s, r; x) = P_0(a_1, b_0; x), \qquad \tilde{P}_n(s, r; x) = \frac{1}{a_1}P_n(a_1, b_0; x), \qquad s = \frac{1}{a_1}, \quad r = \frac{2b_0}{a_1}.$$

The particular case $a_1 = 2$ and $b_0 \neq 0$ corresponds to the co-recursive first kind Chebyshev whose measure follows from (31)

$$\frac{1}{\pi}\frac{\sqrt{1-x^2}}{1+b_0^2-x^2}\,\mathrm{d}x$$

and is of class 2. Let us point out that this result has an obvious misprint in [3, p. 205].

The symmetric case $b_0 = 0$ leads to an example already worked out by Shohat [17] who derived the differential equation. In his notations, with $\sigma = 4(a_1 - 1)/a_1^2$ we have for the measure

$$\frac{2}{\pi}\frac{\sqrt{1-x^2}}{1-\sigma x^2}\,\mathrm{d}x,$$

whose class is 2 and lowers to 0 if $\sigma = 0, 1$.

The class may also lower from 2 to 1 if the parameters are linked by the relations

$$a_1 \pm 2b_0 = 2$$

with the corresponding measures

$$\mu_+ = \frac{1}{2\pi}\sqrt{\frac{1+x}{1-x}}\frac{1}{b_0^2 - (2b_0-1)(1+x)}\,\mathrm{d}x, \quad b_0 < 1,$$
$$\mu_- = \frac{1}{2\pi}\sqrt{\frac{1-x}{1+x}}\frac{1}{b_0^2 + (2b_0+1)(1-x)}\,\mathrm{d}x, \quad b_0 > -1.$$

4.3. Class 2s − 2

Since there are only new examples of SC polynomials with class higher than 2 we shall now consider the simplest symmetric case whose parameters array is

$$\begin{pmatrix} 0 & \underbrace{\begin{matrix} 0 & \cdots & 0 \\ a & \cdots & a \end{matrix}}_{s-1} \end{pmatrix}$$

with $a > 0$. For $a = 1$ we are back to the unperturbed case. One can check that

$$P_n(x) = (\sqrt{a})^n U_n\left(\frac{x}{\sqrt{a}}\right), \quad 0 \leqslant n \leqslant s,$$

which gives for the absolutely continuous part of the orthogonality measure

$$\mu = \frac{\mu_0}{a + (1-a)U_{s-1}^2(x/\sqrt{a})}.$$

The class is therefore $2s - 2$, except if a takes particular values for which the denominator vanishes at $x = \pm 1$. This may happen only for $a > 1$. In these cases the class is lowered to $2s - 4$.

For instance, for $s = 3$, we have class 4 which reduces to 2 for $a = \frac{4}{3}$

$$\mu = \frac{2}{\pi}\frac{\sqrt{1-x^2}}{a + (1-a)(4x^2/a - 1)^2}\,\mathrm{d}x \to \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}(1+3x^2)}\,\mathrm{d}x.$$

The quantities needed for the differential equation are

$$\alpha = (\sqrt{a})^{s-1}\left[U_{s-1}(x)U_{s-1}\left(\frac{x}{\sqrt{a}}\right) - \sqrt{a}U_{s-2}(x)U_s\left(\frac{x}{\sqrt{a}}\right)\right],$$

$$\gamma = (\sqrt{a})^{s-1}\left[-T_s(x)U_{s-1}\left(\frac{x}{\sqrt{a}}\right) + \sqrt{a}T_{s-1}(x)U_s\left(\frac{x}{\sqrt{a}}\right)\right]$$

and also

$$\begin{aligned}\tfrac{1}{2}D_n = a^{s-1}\frac{1-x^2}{1-x^2/a}&\left[sU_s\left(\frac{x}{\sqrt{a}}\right)T_s\left(\frac{x}{\sqrt{a}}\right) - (s+1)U_{s-1}\left(\frac{x}{\sqrt{a}}\right)T_{s+1}\left(\frac{x}{\sqrt{a}}\right)\right]\\ &+ a^sU_s\left(\frac{x}{\sqrt{a}}\right)\left[U_s\left(\frac{x}{\sqrt{a}}\right) - \frac{x}{\sqrt{a}}U_{s-1}\left(\frac{x}{\sqrt{a}}\right)\right] + (n-s)\Delta_s.\end{aligned}$$

Let us observe that, for $a \neq 1$, we have $\deg \Delta_s = 2s - 2$ and $\deg \alpha = 2s - 3$.

Acknowledgements

We are greatly indebted to Theodore Chihara and Waleed Al-Salam for [1] and to Francisco Marcellán for [13, 14]. We would like to thank Pascal Maroni for useful discussions on the semi-classical polynomials.

References

[1] W.R. Allaway, The identification of a class of orthogonal polynomial sets, Thesis, University of Alberta, 1972.

[2] T.S. Chihara, On co-recursive orthogonal polynomials, *Proc. Amer. Math. Soc.* **8** (1957) 899–905.

[3] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).

[4] J. Dombrowski and P. Nevai, Orthogonal polynomials, measures and recurrence relations, *SIAM J. Math. Anal.* **17** (1986) 752–759.

[5] Ja.L. Geronimus, Orthogonal polynomials, *Amer. Math. Soc. Transl. Ser. 2* **108** (1977) 37–130.

[6] E. Hendriksen and H. van Rossum, Semi-classical orthogonal polynomials, in: *Polynômes Orthogonaux et Applications*, Lecture Notes in Math. **1171** (Springer, Berlin, 1985) 354–361.

[7] E.N. Laguerre, Sur la reduction en fractions continues d'une fraction qui satisfait à une équation différentielle du premier ordre dont les coefficients sont rationnels, *J. Math. Pures Appl.* **1** (1885) 135–165; reprinted in *Oeuvres, Vol. 2* (Chelsea, New York, 1972) 685–711.

[8] A.P. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, *J. Comput. Appl. Math.* **57** (1–2) (1995) 215–237 (this issue).

[9] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems of the Special Functions of Mathematical Physics* (Springer, New York, 1966).

[10] F. Marcellán, J.S. Dehesa and A. Ronveaux, On orthogonal polynomials with perturbed recurrence relations, *J. Comput. Appl. Math.* **30** (1990) 203–212.

[11] P. Maroni, Prolégomènes à l'étude des polynômes orthogonaux semi-classiques, *Ann. Mat. Pura Appl.* **4** (1987) 165–184.

[12] P. Maroni, Une théorie algebrique des polynômes orthogaux. Application aux polynômes orthogonaux semi-classiques, in: *Orthogonal Polynomials and their Applications*, IMACS Ann. Comput. Appl. Math. **9** (1991) 95–130.

[13] P. Nevai and W. van Assche, Compact perturbations of orthogonal polynomials, *Pacific J. Math.* **153** (1992) 163–184.

[14] F. Peherstorfer, Finite perturbations of orthogonal polynomials, *J. Comput. Appl. Math.* **44** (1992) 275–302.

[15] A. Ronveaux, Sur l'équation différentielle du second ordre satisfaite par une classe de polynômes orthogonaux semi-classiques, *C. R. Acad. Sci. Paris* **305** (1987) 163–166.

[16] J. Sherman, On the numerators of the convergents of the Stieltjes continued fractions, *Trans. Amer. Math. Soc.* **35** (1933) 64–87.

[17] J. Shohat, A differential equation for orthogonal polynomials, *Duke Math. J.* **5** (1939) 401–417.